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NOTE

REMARKS ON A PAPER OF HIRSCHFELD CONCERNING RAMSEY NUMBERS

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It is shown that a construction of Hirschfeld, which yields lower bounds for a certain class of Ramsey numbers, may be combined with a construction of Erdös, Hajnal and Rado, so as to obtain better bounds.

Ramsey's Theorem asserts that to each triple of positive integers $k, l, r, l \geqslant k$, there corresponds a least integer $n = R(k, l, r)$ such that for every coloring of the k-subsets of an n-set S in r colors there results a homogeneous l-set. In [2], Hirschfeld obtained the following lower bound for R:

$$R(k, l, r^{2r} + 1) \geqslant (l-1)(e_{k+1}^2(r) + 1) + 1, \tag{1}$$

where $e_1(r) = r$ and $e_{i+1}(r) = r^{e_i(r)}$ for $i \geqslant 1$. This bound was established by proving that for $r \geqslant 2$, $\alpha \geqslant 2$, $p + s < l$,

$$R(p+s, l, e_\alpha^2(r) + 1) \geqslant R(p, l, e_{\alpha+s}^2(r) + 1), \tag{2}$$

and then setting $\alpha = 2$, $p = 1$, and $s = k - 1$.

The bound given by (1) grows extremely rapidly with k and r, but only linearly with l. The object of this note is to point out how an even larger lower bound may be obtained. If we set $p = 3$, $s = k - 3$ and $\alpha = 2$ in (2) we get

$$R(k, l, r^{2r} + 1) \geqslant R(3, l, e_{k-1}^2(r) + 1). \tag{3}$$

The advantage in doing this is that instead of starting the iteration of (2) at $p = 1$ (the pigeon hole principle stage) we may now combine (3) with the following results: For $r \geqslant 2$, $l \geqslant 4$,

$$R(3, l, 2r) - 1 \geqslant 2^{R(2, l-1, r) - 1} \tag{4}$$

$$R(3, l, 2r + 1) - 1 \geqslant (l-1)^{R(2, l-1, r) - 1}. \tag{5}$$

A proof of (4) can be found in the paper of Erdös, Hajnal and Rado [1]. (5) does not seem to be in the literature, but it may be established by an argument which is similar to that used to prove (4). We indicate how it goes. Put $m = R(2, l-1, r) - 1$. Let $S = \{1, 2, \ldots, m\}$ and let C be a coloring of S^2 in colors

$c_1, c_2, \ldots, c_r$ which avoids a homogeneous $(l-1)$-set. Let V be the set of $(l-1)^m$ ordered m tuples $(a_1, a_2, \ldots, a_m)$ of integers, where $1 \leq a_i \leq l-1$ for each i. If $u=(a_1, a_2, \ldots, a_m)$ and $v=(b_1, b_2, \ldots, b_m)$ are distinct members of V we write $u>v$ if $a_j>b_j$ for some j and $a_i=b_i$ for $i<j$. We call j the first place where u and v differ. Color V^3 in $2r+1$ colors $d_1, d_2, \ldots, d_{2r+1}$ according to the following scheme. Let $\{u, v, w\} \in V^3$, $u>v>w$. Let j_1 be the first place where u differs from v and j_2 the first place where v differs from w. If $j_1 \neq j_2$ and if $\{j_1, j_2\}$ is colored c_q by C, color $\{u, v, w\}$ in color d_q if $j_1<j_2$ and in color d_{q+r} if $j_1>j_2$. If $j_1=j_2$, color $\{u, v, w\}$ in color d_{2r+1}. Let $A=\{v_1, v_2, \ldots, v_l\}$ be an l-subset of V, $v_1>v_2>\cdots>v_l$, and let j_i denote the first place where v_i differs from v_{i+1}. Suppose A is homogeneous. We must have $j_1<j_2<\cdots<j_{l-1}$ or $j_1>j_2>\cdots>j_{l-1}$ or $j_1=j_2=\cdots=j_{l-1}=j$. However, in the first two cases, we get an $(l-1)$-subset of S, namely $\{j_1, j_1, \ldots, j_{l-1}\}$, which is homogeneous under C. The third case is also impossible since there are only $l-1$ possible values for the jth components of $v_1, v_2, \ldots, v_l$. This establishes (5).

To give some idea of the improvement we note that from (1) we get, on setting $r=2$,

$$R(3,4,17) \geq 3(2^{32}+1)+1.$$

It follows from a result of Fredericksen [2] that $R(2,3,8) \geq 4253$, so that from (5) we get

$$R(3,4,17) > 3^{R(2,3,8)-1} \geq 3^{4252}$$

Note that in general the lower bounds given by (4) and (5) exhibit second order exponential growth in l, and that the improvement effected by (4) and (5) for $k=3$ carries over, when combined with Hirschfeld's inequality (2), to larger values of k.

References

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